

Emergent Continuum Spacetime from a Random, Discrete, Partial Order¹

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Abstract.

There are several indications (from different approaches) that Spacetime at the Plank Scale could be discrete. One approach to Quantum Gravity that takes this most seriously is the Causal Sets Approach. In this approach spacetime is fundamentally a discrete, random, partially ordered set (where the partial order is the causal relation). In this contribution, we examine how timelike and spacelike distances arise from a causal set (in the case that the causal set is approximated by Minkowski spacetime), and how one can use this to obtain geometrical information (such as lengths of curves) for the general case, where the causal set could be approximated by some curved spacetime.

1. Motivation

One of the main problems that any discrete/combinatoric approach to Quantum Gravity² faces is the so called *inverse problem* [1], which has two sides: (a) How do smooth continuum-like structures emerge from the underlying discrete dynamics and (b) how would we recognize that our discrete entity happens to be something that is smooth and continuum-like. Here we will focus on the second part of the inverse problem, for the specific approach to quantum gravity known as *causal sets* (see Ref. [2] or Refs. [3, 4] for reviews).

¹ Based on talk given by P. Wallden at the NEB XIII conference

² Loop quantum gravity, spin foams, causal dynamical triangulations, causal sets, etc.

2. This paper

This contribution is largely based on Ref. [5], where detailed numerics and other related work can be found. In Section 3 we introduce the causal sets approach to Quantum Gravity, in Section 4 we review previous attempts to recover continuum properties from a causal set, in Section 5 we introduce the new suggestion to recover spatial distance in a “flat” causal set and in Section 6 we mention how this can be used for a general (curved spacetime) causal set. We summarize and conclude in Section 7.

3. Causal Sets

Causal sets is a fundamentally discrete approach to quantum gravity. There are several indications that spacetime at the Plank scale could be discrete. These involve the finite black hole entropy, the infinities in general relativity and quantum field theory, as well as indications from other approaches (the discrete volume spectrum in loop quantum gravity, effective minimum length due to dualities in string theory, and others). Causal sets take these indications literally and start with a spacetime that is not a differentiable manifold, but rather a locally finite partially ordered set. More specifically, a causal set is:

- (i) Partially Ordered Set: a set \mathcal{P} with relation \prec , such that $\forall x, y, z \in \mathcal{P}$:
 - (a) $x \prec y$ and $y \prec z \Rightarrow x \prec z$: Transitivity
 - (b) $x \not\prec x$: Irreflexive (with (a) this forbids closed loops)
- (ii) Locally Finite: $\{z | x \prec z \prec y\}$ is a finite set for all $x, y \in \mathcal{P}$.
- (iii) The elements of the set correspond to spacetime points, the order relation (\prec) is the causal relation between spacetime points, and the number of elements corresponds to the spacetime volume.

The second condition is what makes the causal set approach a discrete approach to quantum gravity, since it guarantees that in a finite volume there will be only be a finite number of elements.

A theorem due to Malament (see Ref. [6]) states that one can recover the conformal metric of a spacetime (Lorentzian manifold) using solely the causal (partial) order. This means that all degrees of freedom are encoded in the causal structure, apart from a scale factor. However if we assume spacetime to be discrete, we can fix the scale factor, by counting the number of elements (corresponding to spacetime volume). This leads us to the central conjecture of causal sets “Hauptvermutung”:

Two distinct, non-isometric spacetimes cannot arise from a single causal set.

In order to be precise, we need to define when we say that a causal set is approximated accurately by a spacetime manifold. For this we define the concept of *faithful embedding*:

A faithful embedding is a map ϕ from a causal set \mathcal{P} to a spacetime \mathcal{M} that:

- (i) preserves the causal relation (*i.e.*, $x \prec y \iff \phi(x) \prec \phi(y)$) and
- (ii) is “volume preserving”, meaning that the number of elements mapped to every spacetime region is Poisson distributed, with mean the volume of the spacetime region in fundamental units, and
- (iii) \mathcal{M} does not possess curvature at scales smaller than that defined by the “intermolecular spacing” of the embedding (discreteness scale).

Thus the central conjecture reads as “a causal set cannot be faithfully embedded into two non-isometric spacetimes”. One would like to use these causal sets as fundamental entities in order to construct a quantum theory of gravity. Work on the possible dynamics and phenomenology has been carried out recently (see Refs. [7, 8, 9] and Refs. [10, 11] respectively). However, this contribution deals with some kinematical questions, and in particular with the question “how can one derive effective continuum properties such as spatial distance starting from a causal set, when this causal set can be approximated by a spacetime”. This would arguably, among other things, help us prove the central conjecture.

Before moving to the main topic of this contribution, let us make a remark about Lorentz invariance versus discreteness (see Ref. [12]). It is well known that these two concepts are not (easily) compatible. To have Lorentz invariance, we would like to have (approximately) equal number of elements for every volume of equal size (number-volume correspondence) and this should hold independently of frames. In particular, let us consider a regular lattice approximating a 2-dimensional Minkowski spacetime (see left side of Figure 1). The number-volume correspondence seems to hold, however if we consider the same in a boosted frame (right side of Figure 1), we can clearly see that there are some big “voids” and the lattice cannot be thought of as being approximated by Minkowski spacetime.

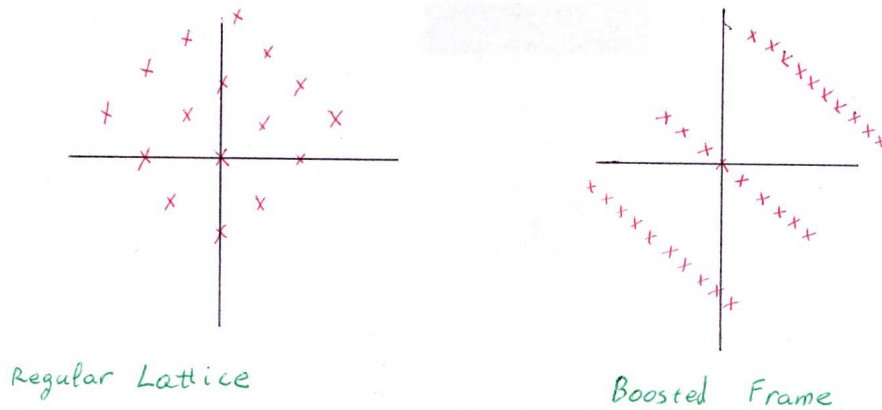


Figure 1. Two dimensional Minkowski spacetime (a) Left: Regular lattice in rest frame (b) Right: Same lattice in a boosted frame

It can be shown that the unique way³ to achieve discreteness and Lorentz invariance, is by considering a *Poisson sprinkling*, i.e. sprinkle randomly elements in the spacetime in question, with probability determined by the poisson distribution with average number of elements proportional to the size of the spacetime volume of each region.

$$P(n) = \frac{(\rho V)^n e^{-\rho V}}{n!}, \quad (1)$$

where ρ is the density, usually fixed at one per Plank volume. Therefore, we can say that a causal set is a *random*, discrete, partially ordered set.

³ Subject to some mathematical subtlety, which allows for some more essentially same solutions

4. Previous Attempts

Much has been written about causal set kinematics (see Refs. [13, 14, 15, 16, 17, 18]); here we will focus on a way to achieve timelike and spacelike distances for a causal set that arises from a sprinkling in a Minkowski spacetime. Moreover we will consider how (some of) these can be generalized for curved spacetimes, such as determining the length of curves.

4.1. Timelike distance

It can be shown that the concept of “timelike” distance (or else the proper time between two timelike separated elements), can be easily defined for a “Minkowski” causal set and can also be generalized for a “curved” causal set. First let us define a *link*. It is the most basic relation between elements, a relation that cannot be deduced by transitivity:

$$x \prec y \text{ are linked if } \nexists z \mid x \prec z \prec y \quad (2)$$

A chain is a collection of elements C such that for all $x, y \in C$ $x \prec y$ or $y \prec x$. Proper time $d(x, y)$, between two related elements $x \prec y$, we define to be (following Ref. [13]) the number of links L in the *longest* chain between (and including) x and y ⁴:

$$d(x, y) := L. \quad (3)$$

In other words, one considers all chains starting from x and ending at y , and counts the number of links in a largest one⁵. This definition is intrinsic to the causal set, and does not depend on whether it can be faithfully embedded into a manifold, nor on the expected dimension of such a manifold. In [13] (and references therein) it is shown that, in the case of a casual set \mathcal{P} which arises by a sprinkling of density ρ into d -dimensional Minkowski space (so one gets a faithful embedding $\phi : \mathcal{P} \rightarrow M$), the distance $d(x, y)$ is proportional to the proper time between the endpoints $\phi(x)$ and $\phi(y)$. In particular the authors state that

$$L(\rho V)^{-1/d} \rightarrow m_d \text{ as } \rho V \rightarrow \infty, \quad (4)$$

for some constant m_d which depends upon the dimension. Here V is the spacetime volume of the causal interval $J^+(x) \cap J^-(y)$ ($J^\pm(x)$ represents the causal future/past of x respectively). The exact value of m_2 is known to be 2, while for other dimensions some bounds exist: $1.77 \leq m_d \leq 2.62$. Note that this expression holds at the limit where ρV goes to infinity, however numerical analysis for the behavior before the asymptotic limit can be found in Ref. [5].

4.2. Naive spatial distance

In the continuum, for Minkowski spacetime, one can show the following: For two spacelike points x, y , their distance is identically the same as the minimum timelike-distance between a point $z \in J^+(x) \cap J^+(y)$ and a point $w \in J^-(x) \cap J^-(y)$, in other words the shortest time taken to go from the common past to the common future. Inspired by this one can define naive spatial distance on a causal set⁶, to be:

$$d_{ns}(x, y) := \min d(w, z) \text{ where } w \in J^-(x) \cap J^-(y) \text{ and } z \in J^+(x) \cap J^+(y). \quad (5)$$

⁴ Note that the Lorentzian character of the partial order is manifested in this definition. Graphs (of ‘finite valence’), on the other hand, naturally embed into Euclidean spaces, and hence one generally defines distance in terms of *shortest* paths on a graph.

⁵ Typically there exist more than one biggest chain.

⁶ This proposal had been made in the past (e.g. Ref. [13]), but was rejected by the original authors themselves, for essentially the same reasons that we present below.

It is clear that for *any* two unrelated elements x, y , the minimum possible distance is 2, when the relevant interval (w, z) is empty, and we shall call this the *trivial distance*. While the definition of Eq. (5) works for causal sets faithfully embedded into \mathbf{M}^2 , it fails for higher dimensions (as first noted in Ref. [13] and numerically confirmed in Ref. [5]). We will explore the reasons for the failure so that we can then proceed to a new proposal that does not suffer from those problems.

4.3. Failure of naive spatial distance

We first consider the continuum. In 1+1 dimensions, there exists a unique pair of elements (w, z) from the common past to the common future of the spacelike separated points in question (x, y) , such that the distance is minimum. The pair of points has w at the point of intersection of the past lightcones of x and y , and similarly z the point of intersection of the future lightcones.

In 2+1 dimensions, the locus of points of intersection of the two past lightcones forms a hyperbola (a one dimensional object). For *each* of the points of the intersection of the past lightcones (w) , there exist a unique point in the intersection of the future lightcones (z) , such that the timelike distance between the pair (w, z) is minimum (and thus equal to the spacelike distance between our target points x, y). We name such pairs *minimizing pairs* and (in the continuum) there are clearly infinite of those. Moreover, we can select a subset of those pairs (of infinite cardinality), such that the intervals between any two of those pairs has arbitrarily small overlap (we fix some small threshold ϵ). We name this collection of minimizing pairs *independent minimizing pairs* (IMP).

Now we return to the causal set that can be faithfully embedded in 2+1 dimensional Minkowski spacetime. It is not difficult to see that there will be elements of the causal set close⁷ to each of the IMP. Each of these pairs, gives (on average) the correct distance, but since the causal set arises as a poisson distribution, there is a finite (but very small) chance that each pair will give a trivial distance. In other words there is a finite probability for each (w, z) IMP, that there will be no elements between w and z . Since we have infinite of those, we are guaranteed that we will find *at least* one IMP, that has no elements between w and z ⁸.

This is a very strong theoretical reason, spotted from the authors of Ref. [13]. However, to check this on a computer is much more difficult, because on a computer we cannot simulate a causal set faithfully embedded into infinite Minkowski spacetime, only a finite region. The *independent* minimizing pairs are widely separated, so testing this effect is difficult, however in Ref. [5] it was clearly observed.

5. Spatial Distance

As we can see, the trouble with the above attempt was that we minimized over all those pairs. Each of those had the correct *average* value, which means that taking an average over suitably chosen pairs would solve this problem. We therefore need:

- (a) A mechanism to select causal set elements which lie close to a minimizing pair. Such a mechanism involves:
 - (i) Finding elements which are close to the intersection of the future (or past) light cones of our unrelated pair of elements.
 - (ii) For each such an element z in the future (say), select an element w in the common past which locates a pair (w, z) which is close to some continuum minimizing pair.

⁷ The precise meaning of “close” is not important here, but to be accurate would require more care.

⁸ The independent condition is required because we need the $d(w, z)$ variables to be independent. Otherwise, we would not be guaranteed to find one pair with zero elements in between.

- (b) To take an **average** over minimizing pairs, and not minimize

For the first issue, we require the definition of a *2-link*: Given unrelated elements x, y of a causal set we define w to be a 2-link (of x, y) if the element w is linked to both x and y (see above for the definition of a link). Given an element x , the elements that are linked to x are elements that lie very close to the lightcone of x . Therefore, it is easy to see that the 2-links lie close to the intersection of the lightcones of x and y . It is easy to see that for dimensions greater than 2, the number of 2-links for any pair of unrelated elements x, y is infinite (corresponding to the (non-countably) infinite points in the continuum). However in 1+1 this is not the case, and most frequently there exist no 2-links at all. In what follows, we could define things slightly differently to account for this problem in 1+1 dimensions⁹. However, (a) we live in 3+1 dimensions and (b) in 1+1 dimensions the naive spatial distance works fine, so we are not going to go into more details on this issue here, and assume the existence of 2-links. We are now in position to define the following procedure that gives us a spatial distance that does not suffer from problems discussed earlier:

- Step 1: Given spacelike elements x, y we find a future 2-link f_i .
- Step 2: Find the element p_i in the common past of x and y that makes the timelike distance $d(p_i, f_i)$ minimum. This will select a minimizing pair.
- Step 3: Store the timelike distance $d^i(x, y)$ for the future 2-link f_i .
- Step 4: Repeat for all other future 2-links.
- Step 5: Take the average over all future 2-links $\langle d^i(x, y) \rangle$ to be the spacelike distance between elements x and y . We call this average the *2-link distance* between x and y .

In Ref. [5] we can see how this distance performs numerically and compare it with naive spatial distance for both the cases that naive spatial distance is valid and where it fails. The results of the simulations agree with the intuitive theoretical expectations analyzed above.

6. Towards Curved Spacetime

In the previous section we considered only how to define spacelike distance between elements of a causal set approximated by a Minkowski spacetime. Using this, we can define a concept of closest spatial neighbor. We define a (symmetric) relation *s-link* between two unrelated elements x, y . This relation exists if the 2-link distance between x and y is less than some fixed threshold λ . This threshold is somewhat arbitrary, but it should be between 2 and 3 (note that with the definition we have above, the smallest conceivable spatial distance would be 2, however taking an average over all the IMP would give a value greater than 2 value no matter how close in the embedding the elements x, y are). Given a target element at the origin of 2+1 dimensional Minkowski spacetime, we can see in Figure 2, that the s-links indeed lie on a hyperboloid as one would expect.

Having defined the concept of an s-link, which is the most elementary relation between unrelated (spacelike) elements, it is straight forward to define what a continuous curve is: It is a collection of elements that can be ordered in such a way that any two consecutive elements are related either as links or as s-links. The length of such continuous curve can simply be derived by counting the number of links and s-links (or else the number of elements).

An important observation is that for spacetimes for which the curvature does not vary rapidly at scales just above the Planck scale, the prescription we define to identify closest

⁹ A concept of closest 2-neighbor that reduces to a 2-link if such thing exists is one obvious attempt.

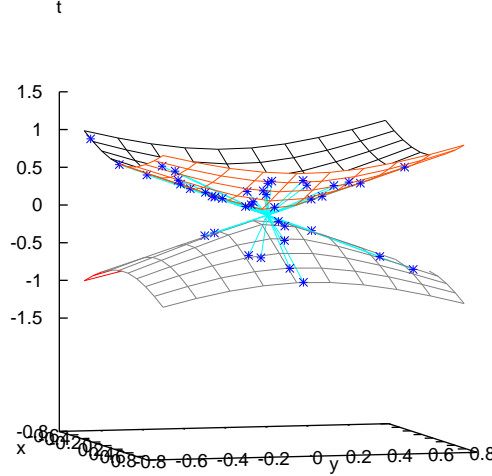


Figure 2. Spatial nearest neighbors of an element in a sprinkling into (a fixed cube in) 2+1 dimensionalowski. $\langle N \rangle = 65\,536$. The future and past light cones of the “origin element” x are shown. The spacelike cyan lines are drawn between x and each neighbor, for emphasis.

neighbors would carry over¹⁰. Identifying closest neighbors in *any* (flat or curved) spacetime means that we can compute lengths of curves in those spacetimes. The very concept of spatial distance is not well defined for curved spacetimes, however the length of curves *is* and the prescription we gave can achieve this.

Detailed numerical analysis, and possible extensions of these ideas in order to recover the full metric, are left for further work. Recovering the full metric may also help in giving direction for deriving quantum dynamics of the causal set (e.g. by rewriting the Einstein-Hilbert action in terms of the relations of the causal set). However, already recovering the lengths of curves is an important achievement in recovering the effective spacetime that approximates our causal set and thus a major step forward in proving the central conjecture.

7. Summary and Conclusions

We have reviewed how to recover timelike distance from a causal set and a failed attempt to do something similar for the spacelike distance on a Minkowski causal set. The reason for this failure was that there exist infinite independent “minimizing pairs” and the definition of naive spatial distance involved minimizing over all of them. Since they turn out to be random variables, minimizing over infinite of those would give a trivial result. To avoid this, we suggested taking an average over those minimizing pairs. However, to do so we need to define what is precisely meant by a minimizing pair in a causal set, which involved the concept of a 2-link. By considering, the closest spatial neighbors, we were able to define length of

¹⁰ It essentially says that for spacetimes that are “locally flat”, in the sense defined above, we can identify which are the closest neighbors using the prescription we mentioned above for flat spacetimes.

curves. Moreover, this relies only on closest neighbors, so for causal sets that do not have curvature at (or close to) the Plank scale, we can hope that the definition of closest neighbors carries over intact. Therefore we can define length of curves in *any* causal set (flat or curved) that can be approximated by a continuum spacetime. Further analysis of the curved spacetime case, and the possibility of fully recovering the metric, is left for future work.

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